Math 10A with Professor Stankova
Worksheet, Discussion #31; Monday, 11/6/2017
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Median and Mean

Example

1. Let $g(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$. Find c such that $f(x) = cg(x)$ is a PDF. Graph f and the CDF F. Find the mean and median of $f(x)$.

> **Solution:** Drawing g, we see that it is a triangle of height 1 and base 2 and so the area under g is 1 so $c = 1$. Then, the CDF F is given by
>
> $$F(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x t dt & 0 \leq x \leq 1 \\ \int_0^1 t dt + \int_1^x (2-t) dt & 1 \leq x \leq 2 \\ \int_0^1 t dt + \int_1^2 (2-t) dt & 1 \leq x \end{cases} = \begin{cases} 0 & x \leq 0 \\ x^2/2 & 0 \leq x \leq 1 \\ \frac{-x^2+4x-2}{2} & 1 \leq x \leq 2 \\ 1 & 1 \leq x \end{cases}$$
>
> The median is when $F(x) = \frac{1}{2}$ and setting them equal gives $x^2/2 = 1/2$ or $x = 1$. The mean is
>
> $$\mu = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 x^2 dx + \int_1^2 x(2-x) dx = \frac{1}{3} + \frac{2}{3} = 1.$$
>
> In this case, our mean and median are the same.

Problems

2. **TRUE** False It is possible for the mean for a discrete PDF to not exist.

> **Solution:** Consider the distribution such that choosing 2^n has probability $\frac{1}{2^n}$. Then the mean doesn't exist, but this is a discrete PDF.

3. **TRUE** False Another name for the mean of a PDF is the expected value.

4. True **FALSE** For a discrete PDF, we can always choose the mean with nonzero probability.

> **Solution:** Consider rolling a dice. The mean is 3.5, which we cannot roll.

5. True **FALSE** For a discrete PDF, we can always choose the median with nonzero probability.

> **Solution:** Again, consider the dice example. The median is 3.5.

6. True **FALSE** There exists a uniform distribution on all the real numbers.

> **Solution:** We cannot make the total area 1.

7. Let $g(x) = \begin{cases} x^2 & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$. Find c such that $f(x) = cg(x)$ is a PDF. Graph f and the CDF F. Find the mean and median of $f(x)$.

Solution: First we calculate

$$\int_{-\infty}^{\infty} g(x)dx = \int_{-1}^{1} x^2 dx = \frac{2}{3}.$$

Therefore, we must have that $\frac{2}{3}c = 1$ or $c = \frac{3}{2}$. The CDF is

$$F(x) = \begin{cases} 0 & x \leq -1 \\ \int_{-1}^{x} 3/2t^2 dt & -1 \leq x \leq 1 \\ \int_{-1}^{1} 3/2t^2 dt & 1 \leq x \end{cases} = \begin{cases} 0 & x \leq -1 \\ \frac{x^3+1}{2} & -1 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases}.$$

The median is when $F(x) = 1/2$ or when $x^3 + 1 = 1$ which is $x = 0$. The mean is

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_{-1}^{1} 3/2x^3 dx = 0,$$

since $3/2x^3$ is an odd function. So again, the median and mean align.

8. Let $g(x) = \begin{cases} xe^{-x^2} & 0 \leq x \\ 0 & \text{otherwise} \end{cases}$. Find c such that $f(x) = cg(x)$ is a PDF. Graph f and the CDF F. Find the mean and median of $f(x)$.

Solution: First we calculate

$$\int_{-\infty}^{\infty} g(x)dx = \int_{0}^{\infty} xe^{-x^2}dx.$$

u subbing with $u = -x^2$ gives us

$$= \int_{0}^{-\infty} \frac{-e^u}{2}du = \frac{1}{2}.$$

Therefore, we must have that $\frac{1}{2}c = 1$ or $c = 2$. The CDF is

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x 2te^{-t^2}dt & x \geq 0 \end{cases} = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x^2} & x \geq 0 \end{cases}.$$

The median is when $F(x) = 1/2$ or when $1 - e^{-x^2} = \frac{1}{2}$ which is $x = \sqrt{\ln 2}$. The mean is

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{\infty} 2x^2e^{-x^2}dx.$$

9. Let $g(x) = \begin{cases} e^{-x} & -1 \leq x \\ 0 & \text{otherwise} \end{cases}$. Find c such that $f(x) = cg(x)$ is a PDF. Graph f and the CDF F. Find the mean and median of $f(x)$.

Solution: First we calculate

$$\int_{-\infty}^{\infty} g(x)dx = \int_{-1}^{\infty} e^{-x}dx = -e^{-x}|_{-1}^{\infty} = e.$$

So $ec = 1$ and $c = 1/e$. The CDF is

$$F(x) = \begin{cases} 0 & x \leq -1 \\ \int_{-1}^x 1/ee^{-t}dt & x \geq -1 \end{cases} = \begin{cases} 0 & x \leq -1 \\ 1 - e^{-x}/e & x \geq -1 \end{cases}.$$

The median is when $F(x) = 1/2$ or when $1 - e^{-x}/e = \frac{1}{2}$ which is $x = \ln 2 - 1$. The mean is

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_{-1}^{\infty} xe^{-x}/edx = (-xe^{-x} - e^{-x})|_{-1}^{\infty}/e = (e - e)/e = 0.$$

10. Let $g(x) = \begin{cases} \frac{1}{x^4} & x \leq -1 \\ 0 & \text{otherwise} \end{cases}$. Find c such that $f(x) = cg(x)$ is a PDF. Graph f and the CDF F. Find the mean and median of $f(x)$.

Solution: First we calculate

$$\int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{-1} \frac{1}{x^4}dx = \frac{-1}{3x^{-3}}|_{-\infty}^{-1} = \frac{1}{3}.$$

Therefore, $c(1/3) = 1$ so $c = 3$. The CDF is

$$F(x) = \begin{cases} \int_{-\infty}^{x} \frac{3}{t^4}dt & x \leq -1 \\ 1 & x \geq -1 \end{cases} = \begin{cases} \frac{-1}{x^3} & x \leq -1 \\ 0 & x \geq -1 \end{cases}.$$

The median is when $F(x) = 1/2$ or when $-1/x^3 = 1/2$ which is $x = -\sqrt[3]{2}$. The mean is

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{-1} \frac{3}{x^3}dx = \frac{-3}{2}.$$

11. Let $g(x) = \frac{1}{1+x^2}$ for $x \geq 0$ and 0 otherwise. Find c such that $f(x) = cg(x)$ is a PDF. Graph f and the CDF F. Find the mean and median of $f(x)$.

Solution: First we calculate

$$\int_{-\infty}^{\infty} g(x)dx = \int_{0}^{\infty} \frac{1}{1+x^2}dx = \arctan(x)|_0^{\infty} = \frac{\pi}{2}.$$

Therefore, $\pi/2c = 1$ and $c = 2/\pi$. The CDF is

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x \frac{2}{\pi(1+t^2)}dt & x \geq 0 \end{cases} = \begin{cases} 0 & x \leq 0 \\ \frac{2}{\pi}\arctan(x) & x \geq 0 \end{cases}.$$

The median is when $F(x) = 1/2$ or when $2/\pi \arctan(x) = \frac{1}{2}$ which is $x = \tan(\pi/4) = 1$. The mean is

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} \frac{2x}{\pi(1+x^2)}dx = 1/\pi \ln(1+x^2)|_0^{\infty} = \infty,$$

so the mean does not exist.

12. Let $g(x) = \begin{cases} \frac{1}{x^4} & 2 \leq x \\ 0 & \text{otherwise} \end{cases}$. Find c such that $f(x) = cg(x)$ is a PDF. Graph f and the CDF F. Find the mean and median of $f(x)$.

Solution: First we calculate

$$\int_{-\infty}^{\infty} g(x)dx = \int_{2}^{\infty} 1/x^4 dx = -1/(3x^3)|_2^\infty = 1/24.$$

Therefore, $c/24 = 1$ and $c = 24$. The CDF is

$$F(x) = \begin{cases} 0 & x \leq 2 \\ \int_2^x 24/t^4 dt & t \geq 2 \end{cases} = \begin{cases} 0 & x \leq 2 \\ 1 - 8/x^3 & x \geq 2 \end{cases}.$$

The median is when $F(x) = 1/2$ or when $1 - 8/x^3 = \frac{1}{2}$ which is $x = \sqrt[3]{16}$. The mean is

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_{2}^{\infty} 24/x^3 dx = -12/x^2|_2^\infty = 3.$$